Investigating Textbook Presentations of Ratio and Proportion

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Many topics within the middle school mathematics curriculum connect to the concept of proportion. Interpretation of proportion situations and understanding of methods for solving proportion problems provides a structure than can be applied to other related topics. As a major resource for secondary mathematics, the extent to which popular textbooks link proportion-related topics was the focus of this study. Our analysis revealed little connectivity of ideas, confusing definitions and frequently illogical calculations. Questions are raised as to the messages texts send to students.

Introduction

In the middle years mathematics curriculum, many topics of study require proportional reasoning skills. For example, proportional reasoning is required in the study of the geometry of plane shapes, in trigonometry, in applications of percentage, as well as for the usual rate, ratio and proportion applications. According to Lesh, Post and Behr (1988) proportional reasoning is a prerequisite for the further study of mathematics: "Proportional reasoning is the capstone of children's elementary school arithmetic and the cornerstone of all that is to follow" (p. 93-94). The development of proportional reasoning then, can be seen as an important goal of primary school mathematics. However, research indicates the elusiveness of such a goal, as many students struggle with proportion-related topics (Behr, Harel, Post & Lesh, 1992; Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Lo & Watanabe, 1997).

In this paper we firstly provide a background to the teaching and learning of proportional reasoning from the extensive literature that has developed. We then report on part of an on-going investigation into the teaching of ratio, rate and proportion in lower secondary school. The study reported here is a preliminary investigation into the ways ratio and proportion are presented in junior secondary textbooks in an attempt to explain one factor that may contribute to the continuing difficulty middle school students experience with topics based on proportional reasoning.

Background

Proportional Reasoning

The complexity of the proportion concept appears to rest in the extent of prior knowledge required for its meaningful development. As outlined by Post, Behr and Lesh (1988), proportional reasoning:

> ...requires firm grasp of various rational number concepts such as order and equivalence, the relationship between the unit and its parts, the meaning and interpretation of ratio, and issues dealing with division, especially as it relates to dividing smaller numbers by larger ones. (p. 80)

Prerequisite knowledge necessary for proportional reasoning has been suggested by others. English and Halford (1995) stated that "fractions are the building blocks of proportion" (p. 254) and Behr et al. (1992) argued that " the concept of fraction order and

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B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 608-616). Sydney: MERGA. ©2002 MERGA Inc.

equivalence, and proportionality are one component of this very significant and global mathematical concept" (p. 316). Streefland (1985) suggested that "learning to view something 'in proportion' or 'in proportion with' precedes the acquisition of the proper concept of ratio" (p. 83). According to Behr et al., the development of an understanding of ratio and proportion is intertwined with many mathematical concepts, including multiplication, division, fractions and decimals, but the essence of proportional reasoning lies in understanding the multiplicative structure of proportional situations. Multiplicative structure is contrasted to additive structure in being able to view, for example, 4 in relation to 8 as multiplying by 2 rather than adding 4.

As previously stated, research has indicated that students' understanding of proportion is generally poor. As stated by Behr et al. (1992), "there is a great deal of agreement that learning rational number concepts remains a serious obstacle in the mathematical development of children" (p. 300). Several authors have suggested reasons for this state of play. According to Streefland (1985) "ratio is introduced too late to be connected with mathematically related ideas such as equivalence of fractions, scale, percentage" (p. 78). English and Halford (1995) suggested that proportional reasoning is taught in isolation and thus remains unrelated to other topics. Behr et al. (1992) stated that "the elementary curriculum is deficient by failing to include the basic concepts and principles relating to multiplicative structures necessary for later learning in intermediate grades (p. 300).

The Topic of Proportion in the Middle Years Curriculum

In describing the topics of proportion in the middle years mathematics curriculum, Ben-Chaim et al. (1998) outlined the general types of proportional reasoning problems as comparisons of two parts of a whole (e.g., ratio of boys to girls in a class), rate or density problems (e.g., cents per litre, kilograms per cubic metre), and scaling problems (e.g., similar triangles). In Ben-Chaim et al.'s analysis, the solution methods for such problems required either a comparison of two complete ratios (e.g., which one is faster) or the calculation of a "missing value" when the other three values in an equivalent pair of ratios is known. The latter solution method arises from representation of the given ratios as a statement of proportion.

The standard algorithm for proportional situations is the representation of equal ratios, that is ${}^{a}/_{b} = {}^{c}/_{d}$ (Touriniare & Pulos, 1985), or ${}^{a}/_{b} = {}^{c}/_{x}$ where a, b and c are given, and x is the unknown. The standard solution procedure for solving proportion equations is via algebraic means: " cross-multiply and solve for x" (Post, Behr & Lesh, 1988, p. 81) or through rule application: "multiply the two numbers across from one another and divide by the other number" (Robinson, 1981, p. 6). The teaching of either the standard algorithm or the rule, however, appears to be a controversial issue. For example, Hart (1981) stated, "Teaching an algorithm such as ${}^{a}/_{b} = {}^{c}/_{d}$ is of little value unless the child understands the need for it and is capable of using it. Children who are not at a suitable level to the understanding of ${}^{a}/_{b} = {}^{c}/_{d}$ will just forget the formula" (p. 101). Further, Cramer, Post and Currier (1992) stated, the "cross-product algorithm is efficient, [yet] it has little meaning. In fact, it is impossible to explain why one would want to find the product of contrasting elements from two different rate pairs...The cross-product rule has no physical referent and therefore lacks meaning for students and for the rest of us as well" (p. 170).

As an historical interlude, the rule application for proportion situations outlined above (multiply the two numbers across from one another and divide by the other number), is generally known as the Rule of Three. The significance of the Rule of Three is outlined by Swetz (1992):

> The 'Rule of Three', commonly known in its time as the 'Golden Rule' or the 'Merchant's rule' was highly esteemed in the fifteenth and sixteenth century as being a powerful mathematical technique applicable to solve many problem situations. Today this rule would be recognised as a statement of simple proportion involving three quantities from which a fourth must be found. (p. 373)

It is through Swetz's words that the Rule of Three is seen as an ancient, efficient and very "handy" rule for solving proportion equations. However, developing meaning for the rule is an issue in developing conceptual understanding of proportion.

One strategy for giving meaning to the cross-multiply method for solving proportion equations (Rule of Three) has been outlined by Robinson (1981) where the construction of ratio "boxes" to correspond to the information given in a ratio situation is advocated. In a manner similar to the ratio tables advocated by Streefland (1985) and English and Halford (1995), Robinson's ratio boxes are designed to reflect the multiplicative structure inherent in proportional situations. For example, the situation of John catching 2 fish to Jim's 3 would be represented as follows:

John's fish	2
Jim's fish	3

When asked to determine how many fish John would have caught if Jim caught 15, the table would be extended as follows:

John's fish	2	2	2	2	2	=	10
Jim's fish	3	3	3	3	3	=	15

In simplified terms, the table would show the situation as a proportion equation:

2	x
3	15

or even simpler: ${}^{2}/_{3} = {}^{x}/_{15.}$

According to Robinson, exploration of proportional situations in this form will lead children to discover the cross-multiply procedure for themselves.

In the John's Fish/Jim's Fish example given above, the fractional representation of the proportion equation follows from the initial tabular representation. Yet the proportion equation (${}^{2}/_{3} = {}^{x}/_{15}$) is representing something quite different to a part/whole fraction situation. The fractional representation of a proportion situation clearly must be linked and connected to the proportional situation it is representing, but it must also be contrasted to the part/whole fraction meaning.

Although there remains controversy over the explicit teaching of proportion equation-solving procedures, there is general consensus in the literature that the proportion equation must be introduced to students in a meaningful manner, with students provided with experiences to enable them to develop their own solution strategies. As proportional reasoning transcends, connects with, and is based upon many other mathematical concepts, instruction must focus on "connectedness" and "structure and context" which are two of a set of basic principles for the design of teaching (Bell, 1993). As stated by Bell, in a constructivist teaching style, students should be helped to see the links between related mathematical ideas (connectedness). Further, Bell contended, most student do not recognise the common structure that underlies parts of mathematics presented as different topics, and hence do not appreciate that similar solution methods can be applied. With the centrality of the idea of proportional reasoning in the middle years mathematics, there is considerable scope for instruction to focus on helping students recognise the proportional structure of many topics (structure and context). Given that many secondary school teachers follow the prescribed textbook for planning instruction in mathematics (Lianghou & Kaeley, 2000), there is an expectation that secondary mathematics texts will assist students make connections between topics that are often presented in different chapters and units.

The Study

The extent to which popular secondary mathematics texts incorporate Bell's principles of connectedness and structure and context was the focus of this investigation. As part of a wider study into the teaching of proportional concepts in the middle years of schooling, the focus of this part of the study was to investigate the ways that proportional concepts are portrayed in the widely used mathematics textbooks that provide the foundation for much of the mathematics instruction in our schools. In this paper, we report on the contents of one chapter from each of two frequently prescribed textbooks, namely: (A) Atkinson & Ward (1996) and (B) Brodie and Swift (1989). (The two textbooks will be referred to as Text A and Text B respectively throughout this report). The selected chapters from the two texts are entitled "Ratio and Proportion" and "Ratios and Rates" respectively. Exploration of the selected chapters occurred through analysis of the given definitions, worked examples and exercises/problems. The aim was to uncover the underlying pedagogy implicit in the chapters. In particular, we were looking for evidence of explication of the structure of proportional reasoning as well as linking and consistency of approach across topics such as ratio, proportion and rate. The solution methods for the problems, particularly those involving finding the missing value in a proportion equation, were also a focus.

Results

Definition of Ratio

A common practice in mathematics textbooks is to highlight definitions of mathematical terms through the use of text boxes, coloured ink, and so on. In Text A, the presented definition of ratio is as follows:

> A **ratio** is a comparison of two or more quantities of the **same kind** and of the same unit. (p. 221) (their emphasis)

This definition is followed by some examples using the colon notation such as "3 km:5 km". The quantities presented are the same units and show a part:part comparison or a whole:whole comparison. There is no mention that the "comparison" is relative (multiplicative) rather than an absolute (additive) one. Comparison by division is implied in the following statement, with ratio represented as a fraction and a percentage:

> Ratios can be written as $\boldsymbol{a} : \boldsymbol{b}$ (we say 'a is to b') or as a fraction ${}^{a}/_{b}$ (we say 'a over b') or as a percentage.
> Where possible, simplify ratios, so a and b are **natural** (whole) numbers.
> We can write 2:3 or ${}^{2}/_{3}$ or 66.6% to all show exactly the same thing. (p. 221)

It is interesting to note here that there is no specification of the quantities being compared in terms of parts and wholes. The definition given moves from real quantities (3km:5km) to mathematical symbols and manipulation where ratio is represented as a fraction and then as a percent. Implicit in the initial definition and example of ratio is the comparisons of part:part or whole:whole. However, the fraction representation provided shifts meaning from a part:part or whole:whole comparison to a part:whole representation in which students' prior experience and most likely understanding of fractions is based. Therefore, the possibility of confusion for students is apparent as no qualification of the use of fraction *notation* to represent a ratio and the use of percentage (a part out of a whole of 100) to express a ratio is given.

In Text B, no single definition of ratio is provided. Rather, the chapter opens with examples of ratios in use, including mixtures such as juice and 2-stroke fuel. The notations for expressing a ratio are defined using a pictorial example of 8 boys and 10 girls on a beach with the accompanying description, that "this can be stated as: 8 to 10, 8 is to 10, 8:10, $^{8}/_{10}$." In this example, the transition from a statement of ratio in words, to the use of a colon to substitute for the words "is to", to the fractional representation is given, but legitimisation of the part:whole fraction notation to represent a clearly part:part or whole:whole situation is not. Further information about ratio representation is given as follows, with ratios as parts explicitly stated:

> The **order** of a ratio is important. If the ratio of sand to cement in mortar is 3:1, an entirely different mortar would result from a mix containing 1 part sand to 3 parts cement.
>
> Ratios may involve *more than two* parts. An example of this is the mixture for concrete, where sand, gravel and cement are mixed in the ratio 2:4:1. . . .
>
> It is possible to simplify ratios in the same way as fractions.

In the last statement, permission is given for ratios to be expressed and manipulated in the same way as fractions, but no conceptual link is provided. The text then goes on to provide a worked example of how ratios and fractions can be simplified:

> Write the ratio of boys to girls in figure 7.1 in simplest form.
> *Solution* Boys : Girls = 8:10
> = $^{8}/_{10}$ (written as a fraction)
> = $^{\cancel{8}4}/_{\cancel{10}}{}^{5}$ (cancel)
> = 4:5 or $^{4}/_{5}$
> Ratios compare quantities of the same kind. If measurements are compared they must have the same units. (p. 257)

In this case the word "part" is used in the discussion but again there is no distinction drawn between the nature of comparison in a ratio (part:part) and the comparison in a fraction (part:whole). The text uses the words "written as a fraction" and "simplify ratios in the same way as fractions", but does not clarify the distinction between the two. No indication of ways to work with ratios containing more than two parts is given and the subsequent exercises do not contain any examples with more than two parts.

Sharing Quantities

The exploration of ratios in terms of parts and their relationship to the whole features in both texts through "sharing" problems. The unitary method is presented to enable students to solve sharing problems. That is, the component unit value of the ratios are explored in terms of their relation to the whole. The setting out of the examples in the two texts is virtually identical. Both texts use the term "parts" to describe the sharing. An example from Text B follows:

> Share \$42 in the ratio 3 : 4.
> Total number of parts = 3 + 4 = 7
> Each part = \$42 ÷ 7
> = \$6
> So 3 parts = 3 × \$6 = \$18
> And 4 parts = 4 × \$6 = \$24
> Total = \$42 (p. 262)

Proportion Problems

Both texts introduce problems involving two equal ratios with one number unknown. Text B first introduces the idea of equivalent ratios and demonstrates two methods to test whether two ratios are equivalent. The first method involves cross-multiplication: "The cross products are *equal*, so the ratios *are equivalent*" (p. 265). The second method involves reducing each ratio to a ratio to one and seeing if the first numbers are equal (e.g., 1.72:1 and 1.75:1 are not equivalent).

Text B provides two methods for solving a proportion problem with one unknown as in the following example.

> The ratio of boys to girls in a class is 4:5. If there are 15 girls, how many boys are in the class?
> First we write the proportion using the information given.
>
> $$\frac{\text{Number of boys}}{\text{Number of girls}} = \frac{4}{5} \Rightarrow \frac{\text{Number of boys}}{15} = \frac{4}{5}$$
>
> Now give the unknown number a variable name so that it is easier to write.
> Let the number of boys = b. So ${}^{b}/_{15} = {}^{4}/_{5}$
>
> To solve the proportion you could use either method below.
>
> METHOD 1
>
> $$\frac{b}{15} = \frac{4}{5}$$
>
> $\times 3$
>
> Numerator and denominator must be multiplied by the same number.
>
> $$\text{So } b = 4 \times 3 = 12$$
>
> METHOD 2
>
> $$\frac{b}{15} \times \frac{4}{5} \qquad \text{(Cross multiply)}$$
>
> $$5b = 4 \times 15$$
>
> $$b = \frac{4 \times \cancel{15}^{3}}{\cancel{5}^{1}} \qquad \text{(Cancel)}$$
>
> $$b = 12$$
>
> There are 12 boys in the class.

Text A does not discuss equivalent ratios before introducing the following proportion example:

> Four car tyres cost \$240 all together. How many tyres could you buy for \$300?

Steps to follow	Solution 2
1. Write as ratio	$4 \text{ new tyres} : n \text{ new tyres} = \$240 : \$300$
2. Substitute pronumerals	$4 : n = 240 : 300$
3. Write as a fraction	$4 / n = 240 / 300$
4. Solve the equation	$4 / n \times 300n = 240 / 300 \times 300n$ $1200 = 240n$ $5 = n$
5. Write your answer	$\therefore$ five tyres could be purchased

Both books present solutions for this type of problem by writing the ratios in fraction notation and solving for the unknown. Text A uses an algebraic equation-solving method that amounts to multiplying both sides of the equation by a number equal to the lowest common denominator of the two fractions. Method 1 in Text B involves the multiplicative idea of proportion. The students are encouraged to think of the multiplier operating between the two known denominators and then apply that multiplier to the numerators including the unknown. Method 2 uses cross-multiplication and then equation solving, rather than a direct application of the Rule of Three.

Rate

Rate is defined similarly in both texts. The definitions presented below are from Text A and Text B respectively:

> A **rate** is a comparison of quantities of *different* kinds. (p. 268)
>
> A **rate** is a comparison of two quantities of **different kinds**, with **different units**. (p. 230)

Again there is no mention of the idea that this is a comparison by division. Both texts discuss commonly used rate types and their units. Text B then focuses mainly on speed as an example of rate. Speed is defined as distance divided by time and the examples use direct substitution into the formula. There are also a number of examples using the idea of a "ready-reckoner". Graphs are used for some situations and a link is made with direct proportion.

The worked examples on rate in Text A are set out in steps, as shown below.

Simplify each of these rates: (a) 8 kilometres in 2 hours (b) 320 tonne of wheat in eight trucks

Steps to follow	**Solution a**	**Solution b**
1 Write the question	8 km in 2 h	320 t in 8 trucks
2 Change to fraction form	= 8 km / 2h	= 320 t / 8 trucks
3 Simplify if possible	= 4 km / 1 h	= 40 t / 1 truck
4 Write your answer	= 4 km/h	= 40 t/truck (p. 231)

The text reminds students of the definition of ratio and the difference between ratio and rate. It also notes that working with rate is similar to working with ratio:

> As with ratio, you should simplify the rate so that the quantities being compared are natural or whole numbers. (p. 231)

Discussion

The aim of this investigation was to examine the use of the concept of proportional reasoning in teaching students about ratio and rate, with the textbook being the major teaching resource. We were particularly interested in finding if explicit use was made of this concept and how explicitly the ideas in the related topics of ratio, proportion and rate were linked. While the aim was not to criticise the authors of the chosen texts, it is inevitable that criticism is implied in our comments.

Both texts introduce definitions of ratio and rate similar to those used in many other texts. Neither text distinguishes between absolute comparisons (subtraction) and relative comparisons (division) in their definitions of ratio and rate. The use of division becomes apparent through the examples. In the definition of rate, the meaning of comparing two quantities of different kinds is not pursued. All that is implied is that division is used but the meaning of "comparing" kilometres with hours is not discussed. In Text A there is no notion of rate as a measure of change, that is, the proportional idea that as one variable changes, the other changes in a related way. Text B makes use of the idea of ready-reckoners which demonstrates proportionality. However, in the text no explicit mention of this link is made and the idea is not discussed with the definition of rate.

The use of fraction notation to work with ratios, proportions and rates is a feature of both texts. However, neither makes the distinction between ratio, a part:part comparison in most of the applications discussed, and fraction and percentage which are normally part:whole comparisons. The notion of parts and wholes only arises when "sharing" problems are addressed. Also, no reasons for the writing of ratios in fraction notation are provided, being presented as the conventional way of working with these problems.

The use of proportional (multiplicative) thinking is presented in these texts as an aside, rather than the main structural idea unifying the topics. Text B has some explicit use of multiplicative thinking in the work on ready-reckoners and in one method of solving a proportion problem. However, the main solution methods presented for the problems involve solving the proportion equation by cross-multiplication and algebraic methods. The solution method for rate problems does not acknowledge the link with proportion at all, relying entirely on the use of a formula. Neither text makes explicit the proportional structure underlying all the problem types investigated, nor do they capitalise on the possible connections that could be made to help students develop a deeper understanding of the ideas.

Conclusion

This brief analysis of just two chapters within two textbooks reveals the limitations of such texts in their definitions, worked examples and suggested solution procedures for the topics of ratio, rate and proportion. The symbolic representation of proportion situations and subsequent manipulation of numbers within proportion equations provides little meaning, either to the real-context of examples presented, or to prior knowledge of other related mathematics topics (e.g., particularly the topic of fractions and percent). The treatment of the topics of ratio, rate and proportion in these two texts appears to offer little connectedness and does little to expose the predominantly part:part (or whole:whole) structure of proportion situations; a structure that can contextualise to other related topics. The definitions and worked examples taken from the texts illustrate the brevity of background information provided for textbook users, suggesting that such information would be quite unhelpful for students and possibly parents. It could be argued that the confusing nature of information given would reinforce a view of mathematics as a meaningless, rule-dominated and highly-specialised subject, accessible to few people. As a first step to exploring the complexity of the topic of proportion, this study has indicated that school texts appear to be limited in their ability to assist students (and teachers) to develop the proportional thinking skills necessary for the successful and meaningful learning of school mathematics.

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